Oral Candidacy Exam - PL Morse Theory and Finiteness Properties of Groups

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Abstract

I aim to survey the paper ”Morse theory and finiteness properties of groups” written by Bestvina and Brady in 1997, which used PL Morse theory to study the finiteness properties of certain groups and found a counterexample to a long standing question by Bieri about whether groups of type FP are finitely presented.

I will first introduce PL Morse theory, a similar tool to Morse theory, but applied to nice cell complexes as affine polytope complexes. I will give definitions of affine polytope complexes and Morse functions on them. Given such a Morse function, the ascending (descending) link of a vertex can be defined. Then the homotopy type of the preimage of the Morse function can be found by adding coned-off ascending or descending links whenever there is a vertex.

I will next introduce some finiteness properties of groups and their relations. These finiteness properties include whether there is a Eilenberg-Maclane space with finite n-skeleton, whether there is a contractible homologically n-connected space with a nice group action, whether there is a finitely generated projective (or free) partial resolution of a given ring in the group ring module catagory. Some relations are obvious, while some relations are still unknown.

I will then state Bestvina and Brady's results. They first found that, given a group epimorphism $\phi : G \to \mathbb{Z}$, and given a ϕ-equivariant Morse function $f : X \to \mathbb{R}$ on a contractible affine polytope complexes X, if we know certain homology or homotopy conditions on all ascending (descending) links, we can deduce certain finiteness properties of the group $Ker(\phi)$. After they studied the right-angled Artin group G_L of a finite flag complex L, they found the converse is also true in this case. That is, some finiteness properties of the group $H_L = Ker(G_L \to \mathbb{Z})$ are completely determined by the homology or homotopy properties of the ascending (descending) links, which are all isomorphic to L. This will gives us an example of a group of type FP but not finitely presented, as long as we require L to be acylic but not simply connected. This example can be realized by taking L as a spine of the Poincaré homology sphere. Moreover, this special example turns out to be a counterexample either to the Eilenberg-Ganea conjecture (a group of cohomological dimension 2 must have geometric dimension 2), or to the Whitehead conjecture (a connected subcomplex of an aspherical 2-complex is aspheical).

References

[1] Mladen Bestvina. PL Morse theory. *Math. Commun.*, 13(2):149–162, 2008.

[2] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[3] Noel Brady Mladen Bestvina. Morse theory and finiteness properties of groups. *Inventions mathematicae*, 1997.